How to get real-valued solutions from complex eigenvalues:

$\lambda_1 = 1+i \qquad \underline{v}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

So the associated soln is:

$$\underline{x}^{(1)} = e^{(1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^t\left\{\underbrace{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\underline{u}} + i\underbrace{\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\underline{v}}\right\}$$

$$= e^t\{\underline{u} + i\underline{v}\}$$

details in Lec 2 part 2

The second eigenvalue + eigenvector are the complex conjugate of the first:

$\lambda_2 = 1-i \qquad \underline{v}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

plug this into $(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})\underline{v}^{(2)} = \underline{0}$ to check

So the associated soln is:

$$\underline{x}^{(2)} = e^{(1-i)t}\,\underline{v}^{(2)} = e^t e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

use $e^{-it} = \cos t - i\sin t$

$$= e^t(\cos t - i\sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= e^t\begin{bmatrix} \cos t - i\sin t \\ -i\cos t - \sin t \end{bmatrix} = e^t\left\{\underbrace{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\text{this is } \underline{u}} - i\underbrace{\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\text{this is } \underline{v}}\right\}$$

$$= e^t\{\underline{u} - i\underline{v}\}$$

Now, these solutions are complex-valued. We want real-valued solutions.

(We also need solutions to be linearly independent)

$\underline{Q}$: Can we take a linear combination of $\underline{x}^{(1)}$ and $\underline{x}^{(2)}$ to get real-valued solns?

Yes. Let's take

$$\hat{\underline{x}}^{(1)} = \frac{1}{2}\left[\underline{x}^{(1)} + \underline{x}^{(2)}\right]$$

$$= \frac{1}{2}\left[e^t\{\underline{u}+i\underline{v}\} + e^t\{\underline{u}-i\underline{v}\}\right]$$

$$= \frac{e^t}{2}\left[2\underline{u}\right] = e^t\underline{u} = \boxed{e^t\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}}$$

$$\hat{\underline{x}}^{(2)} = \frac{1}{2i}\left[\underline{x}^{(1)} - \underline{x}^{(2)}\right]$$

$$= \frac{1}{2i}\left[e^t\{\underline{u}+i\underline{v}\} - e^t\{\underline{u}-i\underline{v}\}\right]$$

$$= \frac{e^t}{2i}\left[2i\underline{v}\right] = e^t\underline{v} = \boxed{e^t\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}}$$

So now we have two real-valued solutions
Need to check that they are linearly independent.

Calculate the Wronskian:

$$W = \begin{vmatrix} e^t\cos t & e^t\sin t \\ -e^t\sin t & e^t\cos t \end{vmatrix}$$

$$= (e^t\cos t)(e^t\cos t) - (e^t\sin t)(-e^t\sin t)$$

$= e^{2t}\cos^2 t + e^{2t}\sin^2 t$

$= e^{2t}(\cos^2 t + \sin^2 t) = e^{2t} \neq 0$

$W \neq 0$ so $\hat{\underline{x}}^{(1)}$ and $\hat{\underline{x}}^{(2)}$ are linearly independent.

So we can write our general solution:

$$\underline{x}(t) = c_1 \hat{\underline{x}}^{(1)} + c_2 \hat{\underline{x}}^{(2)}$$

$$= c_1 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$